Fourier transform

Guillaume Frèche

Version 1.0

In this lecture, we extend our study of Fourier series on the space $L^2_{T_0}(\mathbb{R},\mathbb{C})$ to the more general set $L^2(\mathbb{R},\mathbb{C})$. The idea is to consider aperiodic signals as signals with infinite period.

1 Definition and first properties

Definition 1.1 (Fourier transform)
The **Fourier transform** is the mapping $\mathcal{F}$ from $L^2(\mathbb{R},\mathbb{C})$ onto $L^2(\mathbb{R},\mathbb{C})$ which to any signal $x \in L^2(\mathbb{R},\mathbb{C})$ assigns signal $X \in L^2(\mathbb{R},\mathbb{C})$ defined by:

$$\forall \omega \in \mathbb{R} \qquad X(\omega) = \mathcal{F}(x)(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-i\omega t}dt$$

Signal $X = \mathcal{F}(x)$ is the **Fourier transform** or **spectrum** of x.

Remark:

- As stated in this definition, Fourier transform X is generally a complex-valued function, even if signal x is real-valued. As a consequence, the spectrum is usually represented by two figures, one for the magnitude $|X(\omega)|$ of X, and one for the argument or phase $\text{Arg}(X(\omega))$.
- Fourier transform can also be defined as a function of frequency f instead of impulse $\omega = 2\pi f$. In this case, it is expressed as:

$$\forall \omega \in \mathbb{R} \qquad X(f) = \mathcal{F}(x)(f) = \int_{-\infty}^{+\infty} x(t)e^{-i2\pi ft}dt$$

Proposition 1.1
The Fourier transform satisfies the following properties:

(i) linearity: for two signals x and y, and two scalars α and β, $\mathcal{F}(\alpha x + \beta y) = \alpha\mathcal{F}(x) + \beta\mathcal{F}(y)$;

(ii) symmetry: for any signal x, if we denote $\tilde{x} : t \mapsto x(-t)$, then $\mathcal{F}(\tilde{x}) = \widetilde{\mathcal{F}(x)} = \mathcal{F}(x^*)^*$, which also yields $\mathcal{F}(x^*) = \widetilde{\mathcal{F}(x)}^*$;

(iii) scaling: for any $a > 0$ and any signal x, setting $x_a : t \mapsto x(at)$, for any $\omega \in \mathbb{R}$, $\mathcal{F}(x_a)(\omega) = \dfrac{1}{a}\mathcal{F}(x)\left(\dfrac{\omega}{a}\right)$;

(iv) convolution: for any two signals x and y, $\mathcal{F}(x * y) = \mathcal{F}(x)\mathcal{F}(y)$;

(v) pure delay: for any $a \in \mathbb{R}$ and any signal x, $\mathcal{F}(\tau_a(x)) : \omega \mapsto e^{-i\omega a}\mathcal{F}(x)(\omega)$;

(vi) time differentiation: for any signal x, $\mathcal{F}(x') : \omega \mapsto i\omega\mathcal{F}(x)(\omega)$;

(vii) frequency differentiation: for any signal x, setting $y : t \mapsto -itx(t)$, then $\mathcal{F}(y) = (\mathcal{F}(x))'$

(viii) multiplication: for any two signals x and y, $\mathcal{F}(xy) = \dfrac{1}{2\pi}\left[\mathcal{F}(x) * \mathcal{F}(y)\right]$

(ix) multiplication by a complex exponential: for any $\omega_0 \in \mathbb{R}$, $\mathcal{F}(e_{\omega_0}x) = \tau_{\omega_0}(\mathcal{F}(x))$

PROOF : (i) Linearity results from linearity of integration:

$$\forall\omega \in \mathbb{R} \qquad \mathcal{F}(\alpha x + \beta y)(\omega) = \int_{-\infty}^{+\infty} (\alpha x(t) + \beta y(t))e^{-i\omega t}dt = \alpha\int_{-\infty}^{+\infty} x(t)e^{-i\omega t}dt + \beta\int_{-\infty}^{+\infty} y(t)e^{-i\omega t}dt$$
$$= \alpha\mathcal{F}(x)(\omega) + \beta\mathcal{F}(y)(\omega)$$

(ii) By the change of variable $t \mapsto -t$, we get

$$\forall\omega \in \mathbb{R} \qquad \mathcal{F}(\tilde{x})(\omega) = \int_{-\infty}^{+\infty} x(-t)e^{-i\omega t}dt = \int_{-\infty}^{+\infty} x(t)e^{i\omega t}dt = \left(\int_{-\infty}^{+\infty} x^*(t)e^{-i\omega t}dt\right)^* = \mathcal{F}(x)(-\omega) = \mathcal{F}(x^*)(\omega)^*$$

(iii) By the change of variable $t \mapsto at$, for any $\omega \in \mathbb{R}$,

$$\mathcal{F}(x_a)(\omega) = \int_{-\infty}^{+\infty} x(at)e^{-i\omega t}dt = \frac{1}{a}\int_{-\infty}^{+\infty} x(t)e^{-i\frac{\omega}{a}t}dt = \frac{1}{a}\mathcal{F}(x)\left(\frac{\omega}{a}\right)$$

(iv) Let two signals x and y. For any $\omega \in \mathbb{R}$,

$$\mathcal{F}(x * y)(\omega) = \int_{-\infty}^{+\infty} (x * y)(t)e^{-i\omega t}dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(u)e^{-i\omega u}y(t-u)e^{-i\omega(t-u)}dtdu$$

By the change of variable $(t, u) \mapsto (t, u - t)$ and Fubini's theorem, we get:

$$\mathcal{F}(x * y)(\omega) = \left(\int_{-\infty}^{+\infty} x(t)e^{-i\omega t}dt\right)\left(\int_{-\infty}^{+\infty} y(u)e^{-i\omega u}du\right) = \mathcal{F}(x)(\omega)\mathcal{F}(y)(\omega)$$

(v) Let $a \in \mathbb{R}$ and let a signal x. By the change of variable $t \mapsto t - a$

$$\forall\omega \in \mathbb{R} \qquad \mathcal{F}(\tau_a(x))(\omega) = \int_{-\infty}^{+\infty} x(t-a)e^{-i\omega t}dt = e^{-i\omega a}\int_{-\infty}^{+\infty} x(t)e^{-i\omega t}dt = e^{-i\omega a}\mathcal{F}(x)(\omega)$$

(vi) Let $A > 0$ and $\omega \in \mathbb{R}$. An integration by parts yields:

$$\int_{-A}^{A} x'(t)e^{-i\omega t}dt = \left[x(t)e^{-i\omega t}\right]_{-A}^{A} + i\omega\int_{-A}^{A} x(t)e^{-i\omega t}dt = x(A)e^{-i\omega A} - x(-A)e^{i\omega A} + i\omega\int_{-A}^{A} x(t)e^{-i\omega t}dt$$

Since $x \in L^2(\mathbb{R}, \mathbb{C})$, $\lim_{A\to+\infty} x(A) = \lim_{A\to+\infty} x(-A) = 0$, thus by taking the limit,

$$\mathcal{F}(x')(\omega) = \int_{-\infty}^{+\infty} x'(t)e^{-i\omega t}dt = i\omega\int_{-\infty}^{+\infty} x(t)e^{-i\omega t}dt = i\omega\mathcal{F}(x)(\omega)$$

(vii) By the theorem of differentiation under the integral sign,

$$\forall\omega \in \mathbb{R} \qquad \left(\mathcal{F}(x)\right)'(\omega) = \int_{-\infty}^{+\infty} \frac{\partial}{\partial\omega}\left(x(t)e^{-i\omega t}\right)dt = \int_{-\infty}^{+\infty} (-it)x(t)e^{-i\omega t}dt = \int_{-\infty}^{+\infty} y(t)e^{-i\omega t}dt = \mathcal{F}(y)(\omega)$$

(viii) We prove this property after defining the inverse Fourier transform.
(ix) By property (ix) and by the Fourier transform of a complex exponential,

$$\mathcal{F}(e_{\omega_0}x) = \frac{1}{2\pi}\left[\mathcal{F}(e_{\omega_0}) * \mathcal{F}(x)\right] = \frac{1}{2\pi}\left[2\pi\delta_{\omega_0} * \mathcal{F}(x)\right] = \tau_{\omega_0}(\mathcal{F}(x))$$ ■

Remarks:

- Property (iii) shows that an increasing scaling in time corresponds to a decreasing scaling in frequency, and conversely, which is consistent with the fact that frequency is the inverse of time period.
- Property (iv) applied to $\omega = 0$ yields

$$\int_{-\infty}^{+\infty}(x * y)(t)dt = \left(\int_{-\infty}^{+\infty} x(t)dt\right)\left(\int_{-\infty}^{+\infty} y(t)dt\right)$$

  which is the integration property seen in the lecture about convolution
- Property (iv) is very important because it shows that Fourier transform turns convolution into a pointwise multiplication which is easier to work with, which will prove very useful in the study of LTI systems.

Time and frequency integration is a bit more subtle than differentiation. Let a signal x and denote y an antiderivative of x. By the differentiation property, $X(\omega) = i\omega Y(\omega)$, which implies that $X(0) = 0$, thus we can only define the Fourier transform of antiderivatives of zero-mean signal.
Therefore, consider x_1 and x_2 two antiderivatives of the same zero-mean signal x, i.e. $x_1' = x_2' = x$. Then there exists a constant $C \in \mathbb{C}$ such $x_2 = x_1 + C$. By Corollary 2.7, we can write $X_2(\omega) = X_1(\omega) + 2\pi C\delta(\omega)$. Suppose that x_1 is the zero-mean antiderivative of x, i.e. such that $X_1(0) = \int_{-\infty}^{+\infty} x_1(t)dt = 0$. By the differentiation property, we can write $X(\omega) = i\omega X_1(\omega)$, so that $X_1(\omega) = \frac{X(\omega)}{i\omega}$ and $X_2(\omega) = \frac{X(\omega)}{i\omega} + 2\pi C\delta(\omega)$.
We have the same reasoning for frequency integration: if x_1 and x_2 are two signals whose spectra X_1 et X_2 are antiderivatives of zero-mean X, with $\int_{-\infty}^{+\infty} X_1(\omega)d\omega = 0$, then by frequency differentiation, $x(t) = -itx_1(t)$, so that $x_1(t) = \frac{x(t)}{-it} = \frac{ix(t)}{t}$ and $x_2(t) = \frac{x(t)}{-it} + C\delta(t) = \frac{ix(t)}{t} + C\delta(t)$.

Example 1.1
We introduce the **sign function** $s(t) = \text{sgn}(t) = 2\Upsilon(t) - 1$ which is equal to 1 for $t \geq 0$ and -1 for $t < 0$. For any $A > 0$, $\int_{-A}^{A} s(t)dt = 0$, thus $\int_{-\infty}^{+\infty} s(t)dt = 0$. Since $s' = 2\delta$, we deduce its Fourier transform S:

$$S(\omega) = \frac{\mathcal{F}(2\delta)(\omega)}{i\omega} = \frac{2}{i\omega}$$

Since $\Upsilon(t) = \frac{1}{2}s(t) - \frac{1}{2}$, we deduce the Fourier transform of the Heaviside step function, which is a distribution:

$$\mathcal{F}(\Upsilon)(\omega) = \frac{1}{i\omega} - \frac{1}{2}\delta(\omega)$$

With the same reasoning in the frequency domain, we show that if $S(\omega) = \text{sgn}(\omega)$ then $s(t) = \frac{1}{-i\pi t} = \frac{i}{\pi t}$.

Remark: In telecommunications, we use the **Hilbert transform** $\widehat{x}$ of signal x defined by its Fourier transform: $\widehat{X}(\omega) = -i\text{sgn}(\omega)X(\omega)$. It means that in the time domain, signal $\widehat{x}$ is the result of the convolution of x with $\frac{1}{\pi t}$.

Proposition 1.2
Let x a signal and X its spectrum.

(i) If x is real-valued, then X has a Hermitian symmetry: for any $\omega \in \mathbb{R}$, $X(-\omega) = X(\omega)^*$.

(ii) If x is real-valued and even, then X is also real-valued and even.

PROOF : Property (ii) of Proposition 1.1 shows that $\mathcal{F}(\tilde{x}) = \widetilde{\mathcal{F}(x)} = \mathcal{F}(x^*)^*$ and $\mathcal{F}(x^*) = \widetilde{\mathcal{F}(x)}^*$. If x is real-valued, then $x = x^*$, thus $\widetilde{\mathcal{F}(x)} = \mathcal{F}(x)^*$, i.e. for any $\omega \in \mathbb{R}$, $X(-\omega) = X(\omega)^*$.
Moreover, if x is even, then $x = \tilde{x}$, thus $\mathcal{F}(x) = \widetilde{\mathcal{F}(x)} = \mathcal{F}(x)^*$, i.e. $X = \widetilde{X} = X^*$, therefore X is also real-valued and even. ∎

Definition 1.2 (Frequency response)
The **frequency response** of an LTI system L is the Fourier transform of its impulse response $h = L(\delta)$, i.e.

$$\forall \omega \in \mathbb{R} \qquad H(\omega) = \mathcal{F}(h)(\omega) = \int_{-\infty}^{+\infty} h(t) e^{-i\omega t} dt$$

Remark: If signal x is the input of this LTI system L, then the Fourier transform of the corresponding output y is, for any $\omega \in \mathbb{R}$, $Y(\omega) = X(\omega)H(\omega)$, thus we turned time convolution into frequency multiplication.

Example 1.2
We recall the impulse response of the RC circuit introduced earlier:

$$\forall t \in \mathbb{R} \qquad h(t) = \frac{1}{RC} \exp\left(-\frac{t}{RC}\right) \Upsilon(t) = \frac{1}{\tau} \exp\left(-\frac{t}{\tau}\right) \Upsilon(t)$$

The frequency response of this system is then

$$\forall \omega \in \mathbb{R} \qquad H(\omega) = \int_{-\infty}^{+\infty} h(t) e^{-i\omega t} dt = \frac{1}{\tau} \int_{0}^{+\infty} \exp\left(-\frac{t}{\tau}\right) e^{-i\omega t} dt = \frac{1}{\tau} \left[-\frac{e^{-\left(\frac{1}{\tau}+i\omega\right)t}}{\left(\frac{1}{\tau}+i\omega\right)} \right]_0^{+\infty} = \frac{1}{1+i\omega\tau}$$

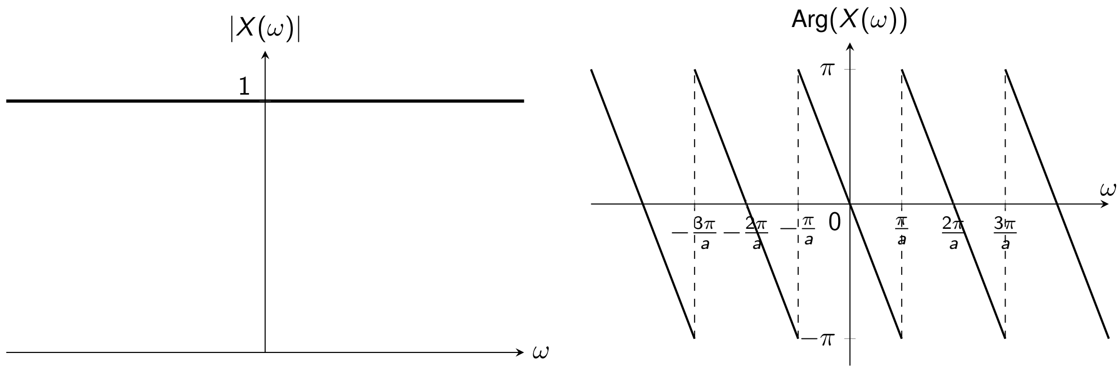
2 Table of Fourier transforms

Proposition 2.1
The Fourier transform of the Dirac delta function $x(t) = \delta$ is the constant function returning 1:

$$x = \delta \qquad \longleftrightarrow \qquad \forall \omega \in \mathbb{R} \qquad X(\omega) = 1$$

In general, for any $a \in \mathbb{R}$, the Fourier transform of Dirac delta function $x(t) = \delta_a(t) = \delta(t-a)$ centered in a is $X(\omega) = e^{-ia\omega}$.

$$x = \delta_a \qquad \longleftrightarrow \qquad \forall \omega \in \mathbb{R} \qquad X(\omega) = e^{-ia\omega}$$



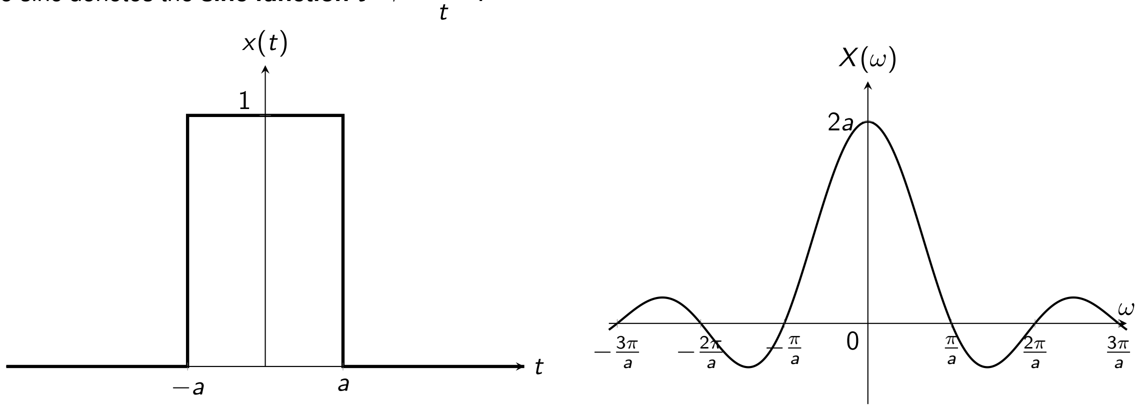
PROOF : For any $\omega \in \mathbb{R}$, $X(\omega) = \displaystyle\int_{-\infty}^{+\infty} \delta(t)e^{-i\omega t}dt = e^{-i\omega 0} = 1$. Then we generalize this result: for $x = \delta_a$, $X(\omega) = e^{-ia\omega}$. ■

Proposition 2.2

Let $a > 0$. The Fourier transform of rectangle signal $x(t) = R_a(t) = \chi_{[-a,a]}(t)$ is

$$\forall \omega \in \mathbb{R} \qquad X(\omega) = \frac{2\sin(a\omega)}{\omega} = 2a\,\mathrm{sinc}(a\omega)$$

where sinc denotes the **sinc function** $t \mapsto \dfrac{\sin(t)}{t}$.



PROOF : For any $\omega \in \mathbb{R}$,

$$X(\omega) = \int_{-\infty}^{+\infty} R_a(t)e^{-i\omega t}dt = \int_{-a}^{a} e^{-i\omega t}dt = \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-a}^{a} = \frac{e^{i\omega a} - e^{-i\omega a}}{i\omega} = \frac{2\sin(a\omega)}{\omega} = 2a\,\mathrm{sinc}(a\omega)$$ ■

Remark: We note $y = x' = R_a'$ the derivative of rectangle signal R_a. We can obtain the Fourier transform of y by two methods. First, for any $t \in \mathbb{R}$, $y(t) = R_a'(t) = \delta_{-a}(t) - \delta_a(t)$. By linearity of Fourier transform, for any $\omega \in \mathbb{R}$,

$$Y(\omega) = \mathcal{F}(\delta_{-a})(\omega) - \mathcal{F}(\delta_a)(\omega) = e^{ia\omega} - e^{-ia\omega} = 2i\sin(a\omega)$$

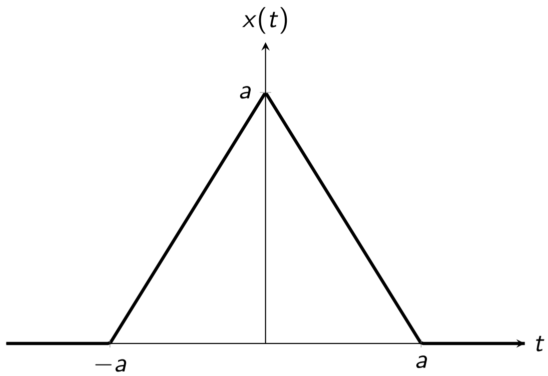
Second, using the Fourier transform of the derivative, from the previous proposition, we have for any $\omega \in \mathbb{R}$,

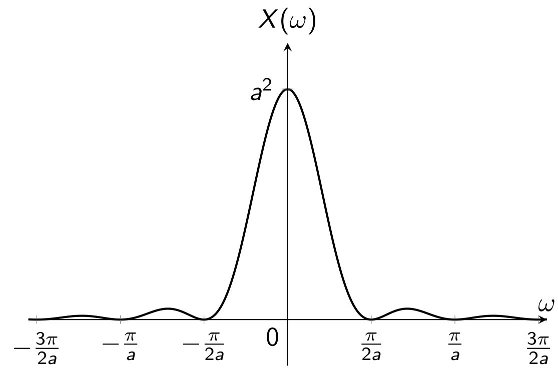
$$Y(\omega) = \mathcal{F}(x')(\omega) = i\omega X(\omega) = i\omega\frac{2\sin(a\omega)}{\omega} = 2i\sin(a\omega)$$

Proposition 2.3

Let $a > 0$. The Fourier transform of triangle signal $x(t) = T_a(t)$ is

$$\forall \omega \in \mathbb{R} \qquad X(\omega) = \frac{4}{\omega^2} \sin^2\left(\frac{a\omega}{2}\right) = a^2 \operatorname{sinc}^2\left(\frac{a\omega}{2}\right)$$





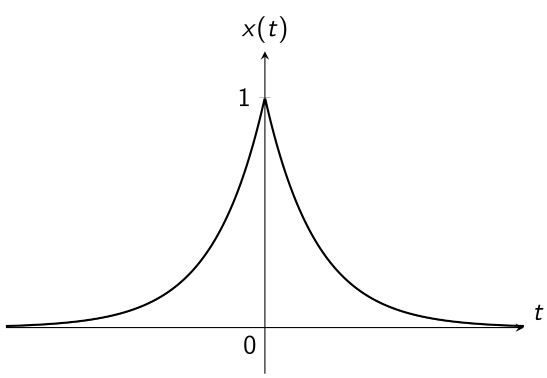
Proof : We have seen in the lecture about convolution that signal T_a is the convolution of rectangle signal $R_{\frac{a}{2}}$ with itself: $T_a = R_{\frac{a}{2}} * R_{\frac{a}{2}}$. Using the Fourier transform of convolution, we get:

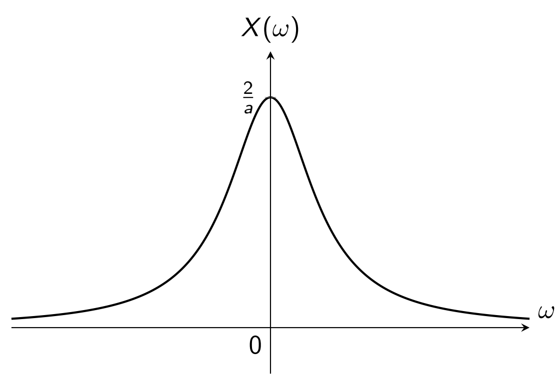
$$\mathcal{F}(T_a)(\omega) = \mathcal{F}(R_{\frac{a}{2}} * R_{\frac{a}{2}})(\omega) = \mathcal{F}(R_{\frac{a}{2}})(\omega)^2 = \frac{4}{\omega^2} \sin^2\left(\frac{a\omega}{2}\right) = a^2 \operatorname{sinc}^2\left(\frac{a\omega}{2}\right)$$ ■

Proposition 2.4

Let $a > 0$. The Fourier transform of exponential $x(t) = e^{-a|t|}$ is:

$$\forall \omega \in \mathbb{R} \qquad X(\omega) = \frac{2a}{a^2 + \omega^2}$$





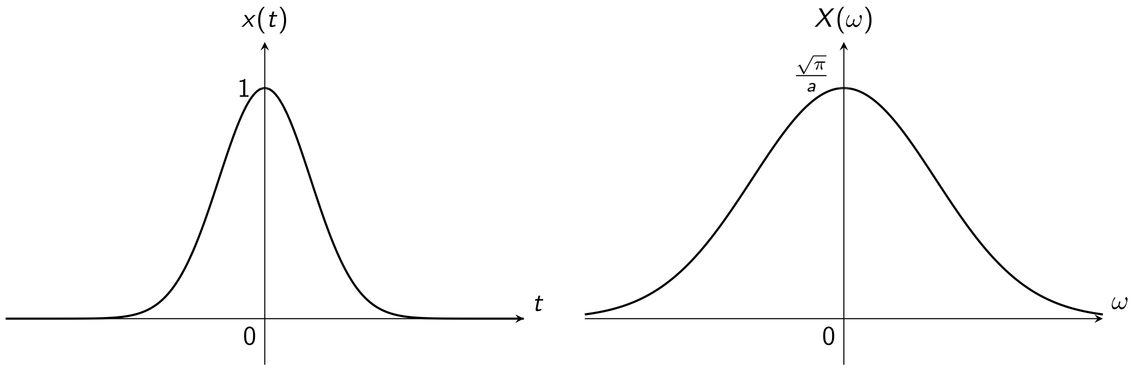
Proof : For $x(t) = e^{-a|t|}$, we have $X(\omega) = \int_{-\infty}^{+\infty} e^{-a|t|} e^{-i\omega t} dt = 2 \int_0^{+\infty} e^{-at} \cos(\omega t) dt$. A double integration by parts gives:

$$\int_0^{+\infty} e^{-at} \cos(\omega t) dt = \frac{a}{\omega^2} - \frac{a^2}{\omega^2} \int_0^{+\infty} e^{-at} \cos(\omega t) dt$$

thus $X(\omega) = \frac{2}{\omega^2} \frac{1}{1 + \frac{a^2}{\omega^2}} = \frac{2a}{a^2 + \omega^2}$. ■

Proposition 2.5
Let $a > 0$. The Fourier transform of gaussian $x(t) = e^{-at^2}$ is:

$$\forall \omega \in \mathbb{R} \qquad X(\omega) = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$$



Proof : We have $X(\omega) = \int_{-\infty}^{+\infty} e^{-at^2} e^{-i\omega t} dt$. Since

$$at^2 + i\omega t = a\left(t^2 + i\frac{\omega}{a}t\right) = a\left(t + i\frac{\omega}{2a}\right)^2 + \frac{\omega^2}{4a}$$

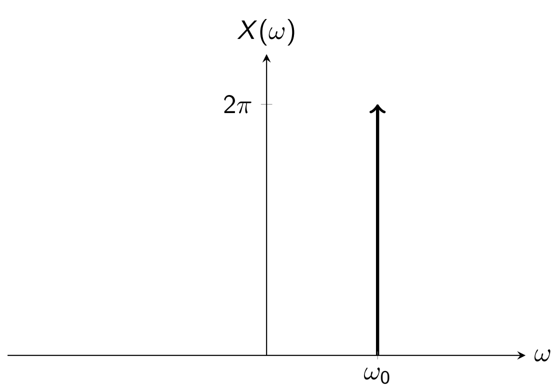
by the change of variable $t \mapsto t - \frac{i\omega}{2a}$ et $t \mapsto \sqrt{a}t$, and by the definition of Gauss' integral $\int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}$, we get

$$X(\omega) = e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{+\infty} \exp\left(-a\left(t + \frac{i\omega}{2a}\right)^2\right) dt = e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{+\infty} e^{-at^2} dt = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} \qquad \blacksquare$$

Remark: It is important to note that the Fourier transform of a gaussian is also a gaussian. This implies that the inverse Fourier transform of a gaussian is a gaussian. This property will be a cornerstone in our definition of the inverse Fourier transform.

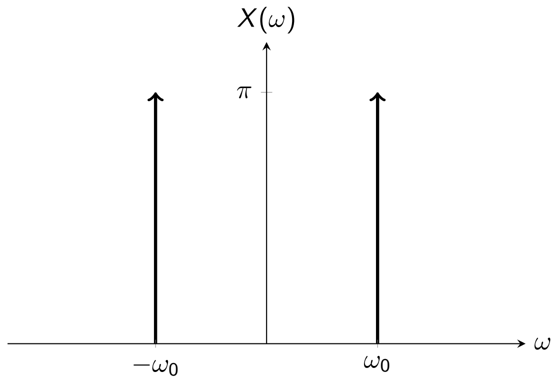
Proposition 2.6
For any $\omega_0 \in \mathbb{R}$, the Fourier transform of complex exponential $x(t) = e^{i\omega_0 t}$ is:

$$\forall \omega \in \mathbb{R} \qquad X(\omega) = 2\pi\delta(\omega - \omega_0)$$



For any $\omega_0 \in \mathbb{R}$, the Fourier transform of cosine $x(t) = \cos(\omega_0 t)$ is:

$$\forall \omega \in \mathbb{R} \qquad X(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



PROOF : This proposition is proved in the next section, once we have introduced the inverse Fourier transform. ■

Corollary 2.7
Using this property with $\omega_0 = 0$, we show that the Fourier transform of constant signal $x : t \mapsto 1$ is $X(\omega) = 2\pi\delta(\omega)$, and by linearity, the Fourier transform of constant signal $x : t \mapsto C$ is $X(\omega) = 2\pi C\delta(\omega)$.

Remark: Let x a periodic signal with period T_0, and let $(c_n(x))_{n\in\mathbb{Z}}$ its Fourier coefficients, i.e. for any $t \in \mathbb{R}$, $x(t) = \sum_{n=-\infty}^{+\infty} c_n(x)e^{in\omega_0 t}$. By linearity of the Fourier transform, spectrum $X = \mathcal{F}(x)$ can be written:

$$\forall \omega \in \mathbb{R} \qquad X(\omega) = \sum_{n=-\infty}^{+\infty} c_n(x)\mathcal{F}\left(e_{n\omega_0}\right)(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} c_n(x)\delta(\omega - n\omega_0)$$

This identity connects Fourier series and Fourier transform. Indeed, a Fourier series is a particular case of discrete Fourier transform represented by Dirac delta functions.

3 Inverse Fourier transform

In this section, we express a time signal $x(t)$ as a function of its spectrum $X(\omega)$, to define the inverse Fourier transform.

Definition 3.1 (Gaussian mollifier)
The **Gaussian mollifier** is the sequence of functions defined for any $n \in \mathbb{N}$ by:

$$\varphi_n : \mathbb{R} \to \mathbb{R} \qquad t \mapsto \frac{n}{\sqrt{\pi}} e^{-n^2 t^2}$$

Lemma 3.1
The limit of sequence $(\varphi_n)_{n\in\mathbb{N}}$ is the Dirac delta function.

PROOF : Denote φ the limit of $(\varphi_n)_{n\in\mathbb{N}}$. Then for any $t \in \mathbb{R}^*$, $\lim_{n\to+\infty} ne^{-n^2t^2} = 0$ thus $\varphi(t) = 0$. Moreover, $\varphi_n(0) = \frac{n}{\sqrt{\pi}}$ thus $\varphi(0) = +\infty$. Finally, we define the following linear forms:

$$\forall n \in \mathbb{N} \quad L_n : x \mapsto \int_{-\infty}^{+\infty} \varphi_n(t)x(t)dt \qquad L : x \mapsto \int_{-\infty}^{+\infty} \varphi(t)x(t)dt$$

Let x be a continuous signal in 0 and bounded over $\mathbb{R}$; and let $\varepsilon > 0$. There exists $\eta > 0$ such that for any $t \in]-\eta,\eta[$, $|x(t)-x(0)| \leq \frac{\varepsilon}{2}$. Then for any $n \in \mathbb{N}$,

$$\begin{aligned}
|L_n(x) - x(0)| &= \left|\int_{-\infty}^{+\infty} \varphi_n(t)x(t)dt - x(0)\right| \leq \int_{-\infty}^{+\infty} \varphi_n(t)|x(t)-x(0)|dt \\
&= \int_{-\eta}^{\eta} \varphi_n(t)|x(t)-x(0)|dt + \int_{|t|>\eta} \varphi_n(t)|x(t)-x(0)|dt \\
&\leq \frac{\varepsilon}{2} + 2\sup_{t\in\mathbb{R}}|x(t)| \int_{|t|>\eta} \varphi_n(t)dt
\end{aligned}$$

Now

$$\lim_{n\to+\infty} \int_{-\eta}^{\eta} \varphi_n(t)dt = \lim_{n\to+\infty} \frac{n}{\sqrt{\pi}} \int_{-\eta}^{\eta} e^{-n^2t^2}dt = \lim_{n\to+\infty} \frac{1}{\sqrt{\pi}} \int_{-n\eta}^{n\eta} e^{-t^2}dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2}dt = 1$$

thus $\lim_{n\to+\infty} \int_{|t|>\eta} \varphi_n(t)dt = 0$, and there exists an index $N \in \mathbb{N}$ such that for any $n \geq N$, $2\sup_{t\in\mathbb{R}}|x(t)| \int_{|t|>\eta} \varphi_n(t)dt \leq \frac{\varepsilon}{2}$ and $|L_n(x) - x(0)| \leq \varepsilon$. Hence sequence $(L_n(x))_{n\in\mathbb{N}}$ converges to $x(0)$, and L is the linear form $x \mapsto x(0)$, corresponding to the Dirac delta function, thus $\varphi = \delta$. ■

Remark: This lemma can be translated in terms of probabilities. Indeed, for any $n \in \mathbb{N}^*$, φ_n is the probability density function of a zero-mean gaussian random variable X_n with standard deviation $\sigma_n = \frac{1}{\sqrt{2}n}$. The sequence of gaussian random variables $(X_n)_{n\in\mathbb{N}^*}$ converges in law to X, a zero-mean gaussian random variable with zero variance, i.e. X is the random variable equal to zero almost surely, whose probability distribution is Dirac delta function.

We express φ_n as a function of its Fourier transform Φ_n by replacing $e^{-i\omega t}$ by $e^{i\omega t}$ in the integral. Using the Fourier transform of a gaussian, we notice that for any $a \in \mathbb{R}$

$$\int_{-\infty}^{+\infty} e^{-a\omega^2} e^{i\omega t} d\omega = \sqrt{\frac{\pi}{a}} e^{-\frac{t^2}{4a}}$$

In particular, taking $a = \frac{1}{4n^2}$,

$$\int_{-\infty}^{+\infty} e^{-\frac{\omega^2}{4n^2}} e^{i\omega t} d\omega = 2n\sqrt{\pi} e^{-n^2t^2} = 2\pi\varphi_n(t)$$

Setting $\Phi_n : \omega \mapsto e^{-\frac{\omega^2}{4n^2}}$, we have

$$\Phi_n(\omega) = \int_{-\infty}^{+\infty} \varphi_n(t)e^{-i\omega t}dt \qquad \varphi_n(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_n(\omega)e^{i\omega t}d\omega \tag{$*$}$$

Lemma 3.2

Let a signal x and its Fourier transform $X = \mathcal{F}(x)$. Then for any $t \in \mathbb{R}$,

$$(\varphi_n * x)(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_n(\omega)X(\omega)e^{i\omega t}d\omega$$

PROOF : By definition of convolution and from (∗),

$$\forall t \in \mathbb{R} \qquad (\varphi_n * x)(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\Phi_n(\omega)e^{i\omega u}d\omega\right)x(t-u)du$$

Then by the change of variable $u \mapsto t-u$,

$$(\varphi_n * x)(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}x(t-u)e^{i\omega u}du\right)\Phi_n(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}x(u)e^{i\omega(t-u)}du\right)\Phi_n(\omega)d\omega$$

Finally,

$$(\varphi_n * x)(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}x(u)e^{-i\omega u}du\right)\Phi_n(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\omega)\Phi_n(\omega)e^{i\omega t}d\omega$$ ∎

Theorem 3.3
Let a signal x and its Fourier transform $X = \mathcal{F}(x)$. Then for any $t \in \mathbb{R}$,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\omega)e^{i\omega t}d\omega$$

With this identity, we define the **inverse Fourier transform** $\mathcal{F}^{-1} : X \mapsto x$.

PROOF : From the previous lemma,

$$\forall t \in \mathbb{R} \qquad (\varphi_n * x)(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Phi_n(\omega)X(\omega)e^{i\omega t}d\omega$$

Since the Dirac delta function is the limit of sequence (φ_n), the dominated convergence theorem implies:

$$\lim_{n\to+\infty}(\varphi_n * x)(t) = (\delta * x)(t) = x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\lim_{n\to+\infty}\Phi_n(\omega)\right)X(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\omega)e^{i\omega t}d\omega$$ ∎

Remark: If we express Fourier transform in terms of frequency f, the relations between a signal and its spectrum become:

$$X(f) = \int_{-\infty}^{+\infty}x(t)e^{-i2\pi ft}dt \qquad x(t) = \int_{-\infty}^{+\infty}X(f)e^{i2\pi ft}df$$

In this case, factor $\frac{1}{2\pi}$ vanishes in the expression of the inverse Fourier transform.
With our definition of inverse Fourier transform, we can now write the missing proofs of the properties in the previous section.
PROOF : (PROPOSITION 1.1, (VIII)) Let two signals x and y and their respective Fourier transforms X and Y. Let $Z = X*Y$ and z the corresponding signal. Then for any $t \in \mathbb{R}$,

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(X*Y)(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}X(u)Y(\omega-u)du\right)e^{i\omega t}d\omega$$

By the change of variable $(u,\omega) \mapsto (u, \omega-u)$ and by Fubini's theorem:

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}X(u)e^{iut}Y(\omega-u)e^{i(\omega-u)t}d\omega du = \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty}X(u)e^{iut}du\right)\left(\int_{-\infty}^{+\infty}Y(\omega)e^{i\omega t}d\omega\right)$$
$$= 2\pi x(t)y(t)$$

By the linearity of the Fourier transform,

$$\mathcal{F}(xy)(\omega) = \frac{1}{2\pi}\mathcal{F}(z)(\omega) = \frac{1}{2\pi}\left[\mathcal{F}(x) * \mathcal{F}(y)\right](\omega)$$ ■

PROOF : (PROPOSITION 2.6) If $X(\omega) = \delta_{\omega_0}(\omega)$, then

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta_{\omega_0}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}e^{i\omega_0 t}$$

thus by linearity of the Fourier transform, $\mathcal{F}(e_{\omega_0})(\omega) = 2\pi\delta_{\omega_0}(\omega) = 2\pi\delta(\omega - \omega_0)$.
Replacing ω_0 by $-\omega_0$, $\mathcal{F}(e_{-\omega_0})(\omega) = 2\pi\delta(\omega + \omega_0)$. Using the linearity of Fourier transform and Euler's identity $c_{\omega_0} = \frac{e_{\omega_0} + e_{-\omega_0}}{2}$, we get $\mathcal{F}(c_{\omega_0})(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$. ■

Theorem 3.4 (Plancherel's identity)
Let a square-integrable signal x and its Fourier transform $X = \mathcal{F}(x)$. Then

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

PROOF : We recognize in the left member the energy of x:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = E(x) = \gamma_x(0) = (x * \tilde{x})(0)$$

with $\tilde{x} : t \mapsto x^*(-t)$. Set $y = (x * \tilde{x})$ and $Y = \mathcal{F}(y)$ its Fourier transform, so that

$$\forall t \in \mathbb{R} \quad y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(\omega)e^{i\omega t}d\omega \qquad \text{et} \qquad E(x) = y(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(\omega)d\omega$$

Applying property (ii) of Proposition 1.1, we can write $\mathcal{F}(\tilde{x}) = \mathcal{F}(x)^*$. Thus we have $Y(\omega) = \mathcal{F}(x * \tilde{x})(\omega) = \mathcal{F}(x)(\omega)\mathcal{F}(\tilde{x})(\omega) = X(\omega)X^*(\omega) = |X(\omega)|^2$. We deduce that:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = y(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$ ■

Remark: Plancherel's identity will be the starter of our discussion about time-frequency duality.